Boundaries

At this point we learned everything and you should in theory be able to solve any global max/min problem.

Caveat: This is not really true, since we will see that things get extra complicated when the domain is not compact. The domains we encountered in Lectures # 17, 18, 19 are (deliberately) all compact. What you learned so far is to solve a global max/min problem provided that the domain is compact.

What you need to know:

- Lagrange critical points we need to use at a domain is determined only by the number of equalities.
- Each equality reduces the dimension of the domain by 1. (unless the equality is redundant)

In particular:

- if # of equalities = # of variables, the domain is a bunch of points, so rather than doing Lagrange multipliers, you should just list all the points of the domain and try them out.
- If # of equalities > # of variables, the domain is going to be empty, so just ignore such domains.

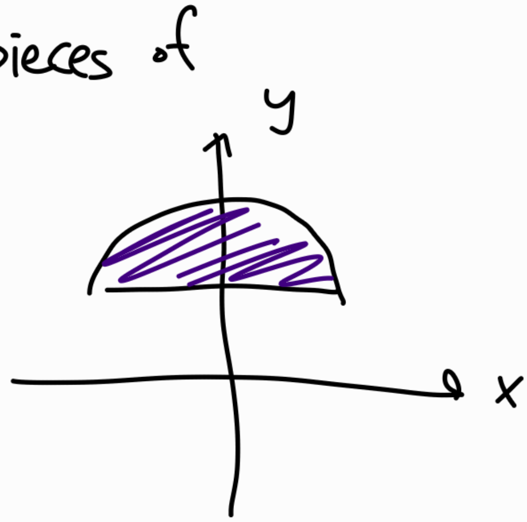
Definition A boundary piece of a domain is the domain formed by transforming some inequalities of the domain equations into equalities.

Ex The non-empty boundary pieces of $\{x^2+y^2 \le 9,\ y \ge 1\}$ are:

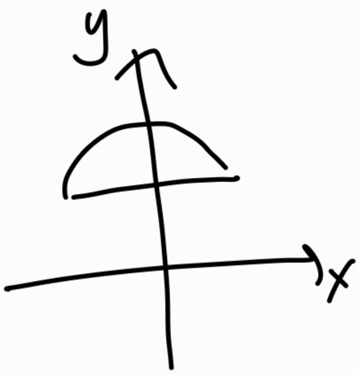
Piece ① $\{x^2+y^2 = 9,\ y \ge 1\}$

Piece ② $\{x^2+y^2 \le 9,\ y = 1\}$

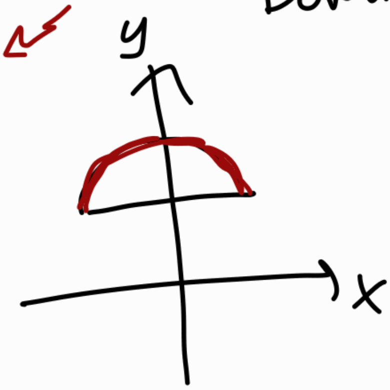
Piece ③ $\{x^2+y^2 = 9,\ y = 1\}$



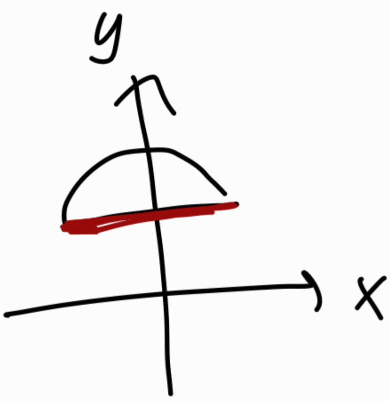
Domain



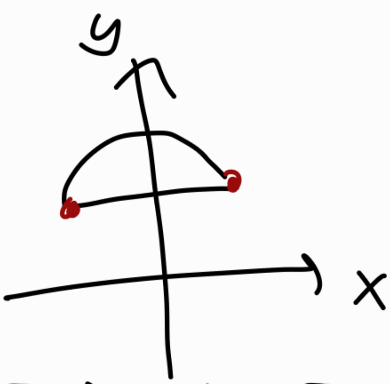
Boundary



Boundary Piece ①



Boundary Piece ②



Boundary Piece ③

Piece ③ is just 2 points $(\sqrt{8},1)$ and $(-\sqrt{8},1)$.

Ex The non-empty boundary pieces of the domain $\{-1 \le x \le 1,\ -1 \le y \le 1\}$ are :

Piece ① $\{-1 = x \le 1,\ -1 \le y \le 1\}$

Piece ② $\{-1 \le x = 1,\ -1 \le y \le 1\}$

Piece ③ $\{-1 \le x \le 1,\ -1 = y \le 1\}$

Piece ④ $\{-1 \le x \le 1,\ -1 \le y = 1\}$

1 equality (Curves)

Empty $\{-1 = x = 1,\ -1 \le y \le 1\}$ ✗

Piece ⑤ $\{-1 = x \le 1,\ -1 = y \le 1\}$

Piece ⑥ $\{-1 = x \le 1,\ -1 \le y = 1\}$

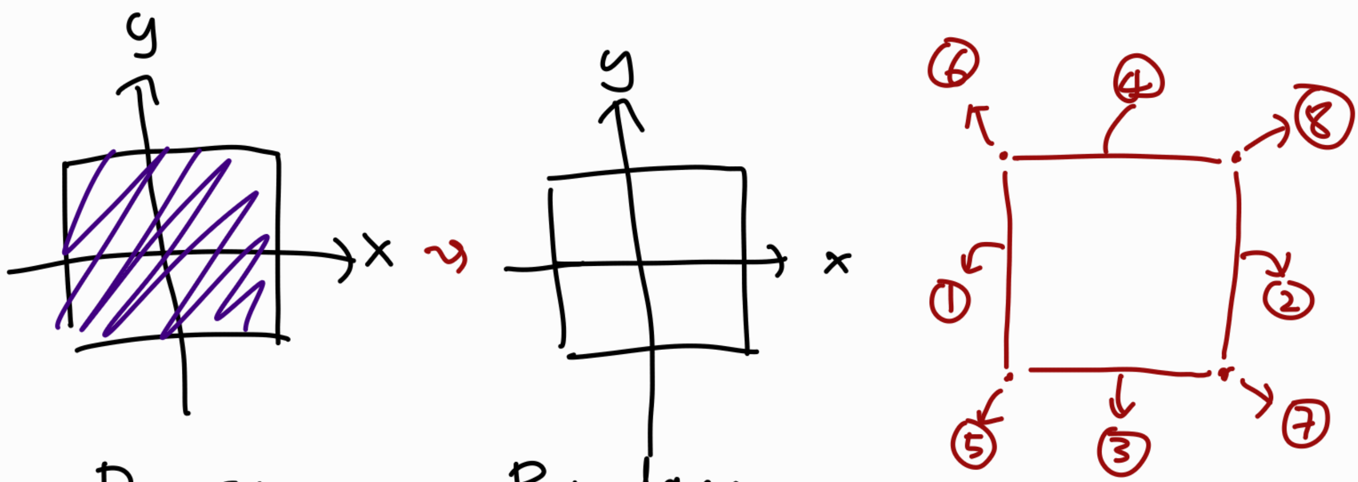
Piece ⑦ $\{-1 \le x = 1,\ -1 = y \le 1\}$

Piece ⑧ $\{-1 \le x = 1,\ -1 \le y = 1\}$

Empty $\{-1 \le x \le 1,\ -1 = y = 1\}$ ✗

2 equalities (Points)

(We already know 3 or more equalities $\Rightarrow$ empty)



Domain

Boundary

Pieces ⑤, ⑥, ⑦, ⑧ are the points $(-1,-1)$, $(-1,1)$, $(1,-1)$, $(1,1)$, respectively.

For the global max/min on a compact domain, the points you have to find are:

* (Lagrange) critical point of the original domain;
* for each non-empty boundary piece, Lagrange critical points on it.
  - As noted earlier, disregard any boundary piece with # of equalities > # of variables
  - For boundary pieces with # of equalities = # of variables, Lagrange critical points are just all those points of the boundary piece. This is OK because such a boundary piece is a bunch of points.

You can regard critical points as Lagrange critical points with 0 equalities, to make the cases more coherent.

I will write down what kinds of points you should find for several different setups.

The inequalities in the sample setup are written all in $\leq$ direction, but the same kinds of rules apply with $\geq$ instead of $\leq$.

Sample Setup ① (2 variables, 2 inequalities)

Global max/min value of $f(x,y)$ on compact domain $\boxed{\{g(x,y)\le k,\ h(x,y)\le l\}}$

Points we look for:

* Critical points (0 equalities) of the original domain $\{g(x,y)\le k,\ h(x,y)\le l\}$
* Lagrange critical points (1 equality) of the boundary piece ① $\{g(x,y)=k,\ h(x,y)\le l\}$
* Lagrange critical points (1 equality) of the boundary piece ② $\{g(x,y)\le k,\ h(x,y)=l\}$
* Lagrange critical points (2 equalities) of the boundary piece ③ $\{g(x,y)=k,\ h(x,y)=l\}$ $\Leftrightarrow$ ALL POINTS IN THIS PIECE (# equalities = # variables = 2)

Types		Equations to solve for (x,y)
Critical points of original domain		$\nabla f(x,y)=\langle 0,0\rangle$ $g(x,y)\le k,\ h(x,y)\le l$
Lagrange critical points of boundary piece ① $\{g(x,y)=k,\ h(x,y)\le l\}$	A	$\nabla g(x,y)=\langle 0,0\rangle$ $g(x,y)=k,\ h(x,y)\le l$
	B	$\nabla f(x,y)=\lambda\nabla g(x,y)$ $g(x,y)=k,\ h(x,y)\le l$
Lagrange critical points of boundary piece ② $\{g(x,y)\le k,\ h(x,y)=l\}$	A	$\nabla h(x,y)=\langle 0,0\rangle$ $g(x,y)\le k,\ h(x,y)=l$
	B	$\nabla f(x,y)=\lambda\nabla h(x,y)$ $g(x,y)\le k,\ h(x,y)=l$
All points of boundary piece ③ $\{g(x,y)=k,\ h(x,y)=l\}$		$g(x,y)=k,\ h(x,y)=l$

<u>Example</u> Find global max/min of $f(x,y) = x^2 - 2y$ on the compact domain $\{x^2+y^2 \le 9,\ y \ge 1\}$.

<u>Sol</u> $g(x,y) = x^2+y^2$, $h(x,y) = y$.

Use the table in the previous page.

* Critical points, original domain $\{g(x,y) \le 9,\ h(x,y) \ge 1\}$

Equations: $\nabla f(x,y) = \langle 0,0\rangle$, $x^2+y^2 \le 9$, $y \ge 1$.

$\nabla f(x,y) = \langle 2x, -2\rangle$, so the y-component is never 0

$\Rightarrow$ $\nabla f(x,y)$ never $\langle 0,0\rangle$

$\Rightarrow$ $\boxed{\text{No points in this case}}$.

* Lagrange critical points, piece 1 $\{g(x,y) = 9,\ h(x,y) \ge 1\}$.

Type A

Equations: $\nabla g(x,y) = \langle 0,0\rangle$, $x^2+y^2 = 9$, $y \ge 1$

$\nabla g(x,y) = \langle 2x, 2y\rangle$, so $\nabla g(x,y) = \langle 0,0\rangle$ only if $x = y = 0$

$\Rightarrow$ Does not satisfy $x^2+y^2 = 9$ $\Rightarrow$ $\boxed{\text{No points in this case}}$.

Type B

Equations: $\nabla f(x,y) = \lambda \nabla g(x,y)$, $x^2+y^2 = 9$, $y \ge 1$.

$\rightsquigarrow$ System of eq. Eq 1 ... $2x = 2\lambda x$

Eq 2 ... $-2 = 2\lambda y$

Eq 3 ... $x^2+y^2 = 9$

Eq 4 ... $y \ge 1$

Both sides of Eq1 are divisible by 2x.

☺ Can divide Eq1 by 2x $\Rightarrow$ $\boxed{1 = \lambda}$ $\xrightarrow{\text{Eq2}}$ $-2 = 2y$ $\Rightarrow$ $\boxed{y = -1}$

But $y = -1$ violates Eq4

$\Rightarrow$ $\boxed{\text{No points in this case}}$.

☹ Cannot divide Eq 1 by 2x because $2x=0$, or $\boxed{x=0}$

$\Rightarrow$ Eq3 $\Rightarrow$ $y^2=9$ $\Rightarrow$ either $y=3$ or $y=-3$.

Combined with Eq4, only $\boxed{y=3}$ is possible

$\Rightarrow$ $\boxed{(0,3)}$ is on the list.

* Lagrange critical points, piece ② $\{g(x,y)\le 9,\ h(x,y)=1\}$

▪ Type Ⓐ

Equations: $\nabla h(x,y)=\langle 0,0\rangle,\ g(x,y)\le 9,\ h(x,y)=1$

Since $\nabla h(x,y)=\langle 0,1\rangle$, it is simply $\neq \langle 0,0\rangle$

$\Rightarrow$ $\boxed{\text{No points in this case.}}$

▪ Type Ⓑ

Equations: $\nabla f(x,y)=\lambda\nabla h(x,y),\ g(x,y)\le 9,\ h(x,y)=1$

$\rightsquigarrow$ System of eq.

Eq1 $\cdots$ $2x=0$

Eq2 $\cdots$ $-2=\lambda$ $\Rightarrow$ $x=0$, $y=1$

Eq3 $\cdots$ $x^2+y^2\le 9$

Eq4 $\cdots$ $y=1$

$\Rightarrow$ $\boxed{(0,1)}$ is on the list.

* All points, piece ③ $\{g(x,y)=9,\ h(x,y)=1\}$

Equations: $x^2+y^2=9,\ y=1$ $\rightsquigarrow$ $x^2=8$ $\rightsquigarrow$ Either $x=\sqrt{8}$ or $x=-\sqrt{8}$.

$\Rightarrow$ $\boxed{(\sqrt{8},1)}$, $\boxed{(-\sqrt{8},1)}$ on the list.

Finally, compare the values of $f(x,y)$ at the points on the list.

Types		(x,y)	$f(x,y)$	Types		(x,y)	$f(x,y)$
Critical, original domain		N/A		Lagrange critical Piece ②	Ⓐ	N/A	
Lagrange critical Piece ①	Ⓐ	N/A			Ⓑ	$(0,1)$	$\boxed{-2}$
	Ⓑ	$(0,3)$	$\boxed{-6}$	All, Piece ③		$(\sqrt{8},1)$	$\boxed{6}$
						$(-\sqrt{8},1)$	$\boxed{6}$

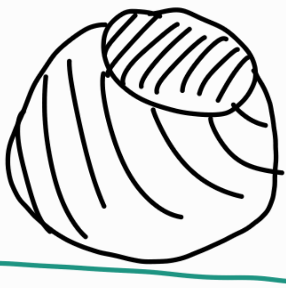
$\Rightarrow$ Global max value: $\boxed{6}$, Global min value: $\boxed{-6}$.

Sample Setup 2 (3 variables, 2 inequalities)

Global max/min value of $f(x,y,z)$ on compact domain $\boxed{\{g(x,y,z)\le k,\ h(x,y,z)\le l\}}$

Types		Equations to solve for (x,y,z)
Critical points, original domain.		$\nabla f(x,y,z)=\langle 0,0,0\rangle$ $g(x,y,z)\le k,\ h(x,y,z)\le l.$
Lagrange critical points Boundary Piece 1 $\{g(x,y,z)=k,\ h(x,y,z)\le l\}$	A	$\nabla g(x,y,z)=\langle 0,0,0\rangle$ $g(x,y,z)=k,\ h(x,y,z)\le l$
	B	$\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$ $g(x,y,z)=k,\ h(x,y,z)\le l$
Lagrange critical points Boundary Piece 2 $\{g(x,y,z)\le k,\ h(x,y,z)=l\}$	A	$\nabla h(x,y,z)=\langle 0,0,0\rangle$ $g(x,y,z)\le k,\ h(x,y,z)=l$
	B	$\nabla f(x,y,z)=\lambda\nabla h(x,y,z)$ $g(x,y,z)\le k,\ h(x,y,z)=l$
Lagrange critical points Boundary Piece 3 $\{g(x,y,z)=k,\ h(x,y,z)=l\}$	A_1	$\nabla g(x,y,z)=\langle 0,0,0\rangle$ $g(x,y,z)=k,\ h(x,y,z)=l$
	A_2	$\nabla h(x,y,z)=\langle 0,0,0\rangle$ $g(x,y,z)=k,\ h(x,y,z)=l$
	B	$\nabla f(x,y,z)=\lambda\nabla g(x,y,z)+\mu\nabla h(x,y,z)$ $g(x,y,z)=k,\ h(x,y,z)=l$

Lagrange critical points of piece 3 do not simplify in this setup because # of variables $=3>2=$ # of equalities.

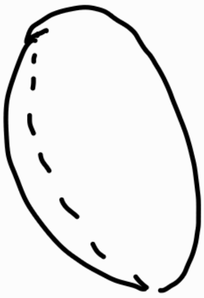


Sample domain

$\{x^2+y^2+z^2\le 6,\ x+y+z\le 4\}$

(Solid sphere w/ a cut)

Example Find global max/min values of $f(x,y,z) = x^2+y^2-z^2$ on the compact domain $\{x^2+y^2+z^2 \le 10,\ x-z \ge 4\}$



Domain
("spherical cap")

Sol $g(x,y,z) = x^2+y^2+z^2$, $h(x,y,z) = x-z$.

Use the table in the previous page.

* Critical points, original domain $\{g(x,y,z) \le 10,\ h(x,y,z) \ge 4\}$

Equations: $\nabla f(x,y,z) = \langle 0,0,0\rangle$, $g(x,y,z) \le 10$, $h(x,y,z) \ge 4$.

$\nabla f(x,y,z) = \langle 2x, 2y, -2z\rangle \Rightarrow \nabla f(x,y,z) = \langle 0,0,0\rangle$ only if $(x,y,z) = (0,0,0)$

$\Rightarrow$ Does not satisfy $x-z \ge 4$

$\Rightarrow$ $\boxed{\text{No points in this case}}$

* Lagrange critical points, piece ① $\{g(x,y,z) = 10,\ h(x,y,z) \ge 4\}$

■ Type Ⓐ

Equations: $\nabla g(x,y,z) = \langle 0,0,0\rangle$, $g(x,y,z) = 10$, $h(x,y,z) \ge 4$.

$\nabla g(x,y,z) = \langle 2x, 2y, 2z\rangle$ is $\langle 0,0,0\rangle$ only if $(x,y,z) = (0,0,0)$ $\Rightarrow$ Does not satisfy $x-z \ge 4$

$\Rightarrow$ $\boxed{\text{No points in this case}}$.

■ Type Ⓑ

Equations: $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$, $g(x,y,z)=10$, $h(x,y,z) \geq 4$

$\rightsquigarrow$ System of eq. Eq1 ⋯ $2x = 2\lambda x$

Eq2 ⋯ $2y = 2\lambda y$

Eq3 ⋯ $-2z = 2\lambda z$

Eq4 ⋯ $x^2+y^2+z^2=10$

Eq5 ⋯ $x - z \geq 4$

Both sides of Eq1 are divisible by $2x$.

▲ ☺ can divide Eq1 by $2x$ and get $\boxed{\lambda=1}$

$\xrightarrow{\text{Eq3}}$ $-2z=2z \Rightarrow \boxed{z=0}$ $\xrightarrow{\text{Eq5}}$ $x \geq 4$.

On the other hand, Eq4 says $x^2+y^2=10 \Rightarrow x^2 \leq 10$.

$\Rightarrow$ Contradiction ($x \geq 4$ and $x^2 \leq 10$ at the same time)

$\Rightarrow$ $\boxed{\text{No points in this case}}$

▲ ☹ can't divide Eq1 by $2x$ because $2x=0$, or $\boxed{x=0}$

$\xrightarrow{\text{Eq5}}$ $-z \geq 4 \Rightarrow z \leq -4$

On the other hand, Eq4 says $y^2+z^2=10 \Rightarrow z^2 \leq 10$

$\Rightarrow$ Contradiction ($z \leq -4$ and $z^2 \leq 10$ at the same time)

$\Rightarrow$ $\boxed{\text{No points in this case}}$

✶ Lagrange critical points, piece ② $\{g(x,y,z) \leq 10,\ h(x,y,z) = 4\}$

■ Type Ⓐ

Equations: $\nabla h(x,y,z) = \langle 0,0,0 \rangle$, $g(x,y,z) \leq 10$, $h(x,y,z)=4$

$\nabla h(x,y,z) = \langle 1,0,-1 \rangle$ is simply $\neq \langle 0,0,0 \rangle$

$\Rightarrow$ $\boxed{\text{No points in this case}}$.

■ Type (B)

Equations: $\nabla f(x,y,z) = \lambda \nabla h(x,y,z)$, $g(x,y,z) \le 10$, $h(x,y,z) = 4$.

⇝ System of eq.
Eq1 ... $2x = \lambda$
Eq2 ... $2y = 0$ ⇨ $\boxed{y=0}$
Eq3 ... $-2z = -\lambda$
Eq4 ... $x^2+y^2+z^2 \le 10$
Eq5 ... $x - z = 4$

Eq1 & Eq3 ⇨ $x = z$ ⇨ Violates Eq5

⇨ No points in this case.

* Lagrange critical points, piece (3) $\{g(x,y,z)=10,\ h(x,y,z)=4\}$

■ Type (A_1)

Equations: $\nabla g(x,y,z) = \langle 0,0,0\rangle$, $g(x,y,z)=10$, $h(x,y,z)=4$.

$\nabla g(x,y,z) = \langle 2x, 2y, 2z\rangle$ is $\langle 0,0,0\rangle$ only if $x=y=z=0$, which does not satisfy $x^2+y^2+z^2=10$

⇨ No points in this case.

■ Type (A_2)

Equations: $\nabla h(x,y,z) = \langle 0,0,0\rangle$, $g(x,y,z)=10$, $h(x,y,z)=4$

$\nabla h(x,y,z) = \langle 1,0,-1\rangle$ is never $\langle 0,0,0\rangle$

⇨ No points in this case.

■ Type (B)

Equations: $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$,
$g(x,y,z)=10$, $h(x,y,z)=4$

⇝ System of eq.

Eq1 ⋯ $2x = 2\lambda x + \mu$

Eq2 ⋯ $2y = 2\lambda y$

Eq3 ⋯ $-2z = 2\lambda z - \mu$

Eq4 ⋯ $x^2+y^2+z^2=10$

Eq5 ⋯ $x - z = 4$

Both sides of Eq2 are divisible by $2y$

▲ ☺ can divide Eq2 by $2y$ and get $\boxed{\lambda=1}$ $\xrightarrow{\text{Eq1}}$

$2x = 2x + \mu \Rightarrow \boxed{\mu=0} \xrightarrow{\text{Eq3}} -2z = 2z \Rightarrow \boxed{z=0} \xrightarrow{\text{Eq5}} \boxed{x=4}$

But $x=4, z=0$ already violates Eq4 because it tells us that $y^2 = 10-16=-6$ which is impossible.

⇒ $\boxed{\text{No points in this case}}$

▲ ☹ cannot divide Eq2 by $2y$ because $2y=0$, or $\boxed{y=0}$

$\xrightarrow{\text{Eq4}} x^2+z^2=10$. Since Eq5 ⇒ $x = z+4$.

$\xrightarrow{\text{together}} (z+4)^2+z^2=10 \Rightarrow 2z^2+8z+16=10$

$\Rightarrow 2z^2+8z+6=0 \Rightarrow z^2+4z+3=0 \Rightarrow (z+1)(z+3)=0$

⇒ Either $\boxed{z=-1}$ or $\boxed{z=-3}$ ⇒ $\boxed{(3,0,-1)}$, $\boxed{(1,0,-3)}$ are points on the list.

$\boxed{x=3}$ $\boxed{x=1}$

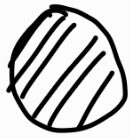
Types		(x,y,z)	f(x,y,z)
Critical original domain		N/A	
Lagrange critical Boundary piece ①	A	N/A	
	B	N/A	
Lagrange critical Boundary piece ②	A	N/A	
	B	N/A	
Lagrange critical Boundary piece ③	A	N/A	
	A	N/A	
	B	(3,0,-1)	$f(3,0,-1) = \boxed{8}$
		(1,0,-3)	$f(1,0,-3) = \boxed{-8}$

⇒ Global max value $\boxed{8}$, Global min value $\boxed{-8}$

I will record various setups and the corresponding table of equations.

Sample setup ③ (2 variables, 1 inequality)

Global max/min of $f(x,y)$ on compact domain $\{g(x,y) \le k\}$

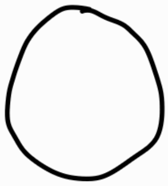


Sample domain
$\{x^2+y^2 \le 1\}$
(Inside of a circle)

Types		Equations to solve for (x,y)
Critical points, original domain		$\nabla f(x,y) = \langle 0,0 \rangle$, $g(x,y) \le k$
Lagrange critical points Boundary Piece ① $\{g(x,y)=k\}$	A	$\nabla g(x,y) = \langle 0,0 \rangle$, $g(x,y) = k$
	B	$\nabla f(x,y) = \lambda \nabla g(x,y)$, $g(x,y) = k$

Sample setup ④ (2 variables, 1 equality)

Global max/min of $f(x,y)$ on compact domain $\{g(x,y) = k\}$



Sample domain
$\{x^2+y^2 = 1\}$
(Edge of a circle)

Types		Equations to solve for (x,y)
Lagrange critical points original domain	A	$\nabla g(x,y) = \langle 0,0 \rangle$, $g(x,y) = k$
	B	$\nabla f(x,y) = \lambda \nabla g(x,y)$, $g(x,y) = k$

Sample setup 5 (2 variables, 1 inequality, 1 equality)

Global max/min of $f(x,y)$ on compact domain $\boxed{\{g(x,y)=k,\ h(x,y)\le \ell\}}$

Sample domain

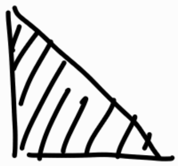
$\{x^2+y^2=1,\ y\ge 0\}$

(Edge of a circle, cut in half)

Types		Equations to solve for (x,y)
Lagrange critical points original domain	A	$\nabla g(x,y)=\langle 0,0\rangle,\ g(x,y)=k,\ h(x,y)\le\ell$
	B	$\nabla f(x,y)=\lambda\nabla g(x,y)$ $g(x,y)=k,\ h(x,y)\le\ell$
All points Boundary piece 1 $\{g(x,y)=k,\ h(x,y)=\ell\}$		$g(x,y)=k,\ h(x,y)=\ell$

Sample setup ⑥ (2 variables, 3 inequalities)

Global max/min of $f(x,y)$ on compact domain $\{g(x,y)\le k,\ h(x,y)\le \ell,\ i(x,y)\le m\}$



Sample domain

$\{x\ge 0,\ y\ge 0,\ x+y\le 1\}$

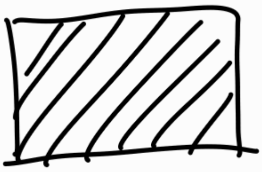
(Inside of a triangle)

Types		Equations to solve for (x,y)
Critical points original domain		$\nabla f(x,y)=\langle 0,0\rangle$ $g(x,y)\le k,\ h(x,y)\le \ell,\ i(x,y)\le m$
Lagrange critical points Boundary piece ① $\{g(x,y)=k,\ h(x,y)\le \ell,\ i(x,y)\le m\}$	A	$\nabla g(x,y)=\langle 0,0\rangle$ $g(x,y)=k,\ h(x,y)\le \ell,\ i(x,y)\le m$
	B	$\nabla f(x,y)=\lambda\nabla g(x,y)$ $g(x,y)=k,\ h(x,y)\le \ell,\ i(x,y)\le m$
Lagrange critical points Boundary piece ② $\{g(x,y)\le k,\ h(x,y)=\ell,\ i(x,y)\le m\}$	A	$\nabla h(x,y)=\langle 0,0\rangle$ $g(x,y)\le k,\ h(x,y)=\ell,\ i(x,y)\le m$
	B	$\nabla f(x,y)=\lambda\nabla h(x,y)$ $g(x,y)\le k,\ h(x,y)=\ell,\ i(x,y)\le m$
Lagrange critical points Boundary piece ③ $\{g(x,y)\le k,\ h(x,y)\le \ell,\ i(x,y)=m\}$	A	$\nabla i(x,y)=\langle 0,0\rangle$ $g(x,y)\le k,\ h(x,y)\le \ell,\ i(x,y)=m$
	B	$\nabla f(x,y)=\lambda\nabla i(x,y)$ $g(x,y)\le k,\ h(x,y)\le \ell,\ i(x,y)=m$
All points, boundary piece ④ $\left\{\begin{matrix} g(x,y)=k \\ h(x,y)=\ell \\ i(x,y)\le m\end{matrix}\right\}$		$g(x,y)=k,\ h(x,y)=\ell,\ i(x,y)\le m$
All points, boundary piece ⑤ $\left\{\begin{matrix} g(x,y)=k \\ h(x,y)\le \ell \\ i(x,y)=m\end{matrix}\right\}$		$g(x,y)=k,\ h(x,y)\le \ell,\ i(x,y)=m$
All points, boundary piece ⑥ $\left\{\begin{matrix} g(x,y)\le k \\ h(x,y)=\ell \\ i(x,y)=m\end{matrix}\right\}$		$g(x,y)\le k,\ h(x,y)=\ell,\ i(x,y)=m$

Sample setup 7 (2 variables, 4 inequalities)

Global max/min of $f(x,y)$ on compact domain

$$\{g(x,y)\le k,\ h(x,y)\le l,\ i(x,y)\le m,\ j(x,y)\le n\}$$



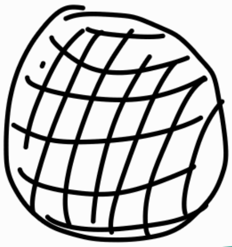
Sample domain
$\{-2\le x\le 2,\ -1\le y\le 1\}$
(Inside of a rectangle)

As you can see here there are usually less than 10 boundary pieces. Namely, some boundary pieces below may be empty.

Types		Equations to solve for (x,y)
Critical, original domain		$\nabla f(x,y)=\langle 0,0\rangle,\ g(x,y)\le k,\ h(x,y)\le l,\ i(x,y)\le m,\ j(x,y)\le n$
Lagrange critical, boundary piece 1 $\{g(x,y)=k,\ h(x,y)\le l,\ i(x,y)\le m,\ j(x,y)\le n\}$	A	$\nabla g(x,y)=\langle 0,0\rangle,\ g(x,y)=k,\ h(x,y)\le l,\ i(x,y)\le m,\ j(x,y)\le n$
	B	$\nabla f(x,y)=\lambda\nabla g(x,y),\ g(x,y)=k,\ h(x,y)\le l,\ i(x,y)\le m,\ j(x,y)\le n$
Lagrange critical, boundary piece 2 $\{g(x,y)\le k,\ h(x,y)=l,\ i(x,y)\le m,\ j(x,y)\le n\}$	A	$\nabla h(x,y)=\langle 0,0\rangle,\ g(x,y)\le k,\ h(x,y)=l,\ i(x,y)\le m,\ j(x,y)\le n$
	B	$\nabla f(x,y)=\lambda\nabla h(x,y),\ g(x,y)\le k,\ h(x,y)=l,\ i(x,y)\le m,\ j(x,y)\le n$
Lagrange critical, boundary piece 3 $\{g(x,y)\le k,\ h(x,y)\le l,\ i(x,y)=m,\ j(x,y)\le n\}$	A	$\nabla i(x,y)=\langle 0,0\rangle,\ g(x,y)\le k,\ h(x,y)\le l,\ i(x,y)=m,\ j(x,y)\le n$
	B	$\nabla f(x,y)=\lambda\nabla i(x,y),\ g(x,y)\le k,\ h(x,y)\le l,\ i(x,y)=m,\ j(x,y)\le n$
Lagrange critical, boundary piece 4 $\{g(x,y)\le k,\ h(x,y)\le l,\ i(x,y)\le m,\ j(x,y)=n\}$	A	$\nabla j(x,y)=\langle 0,0\rangle,\ g(x,y)\le k,\ h(x,y)\le l,\ i(x,y)\le m,\ j(x,y)=n$
	B	$\nabla f(x,y)=\lambda\nabla j(x,y),\ g(x,y)\le k,\ h(x,y)\le l,\ i(x,y)\le m,\ j(x,y)=n$
All points, boundary piece 5 $\{g(x,y)=k,\ h(x,y)=l,\ i(x,y)\le m,\ j(x,y)\le n\}$		$g(x,y)=k,\ h(x,y)=l,\ i(x,y)\le m,\ j(x,y)\le n$
All points, boundary piece 6 $\{g(x,y)=k,\ h(x,y)\le l,\ i(x,y)=m,\ j(x,y)\le n\}$		$g(x,y)=k,\ h(x,y)\le l,\ i(x,y)=m,\ j(x,y)\le n$
All points, boundary piece 7 $\{g(x,y)=k,\ h(x,y)\le l,\ i(x,y)\le m,\ j(x,y)=n\}$		$g(x,y)=k,\ h(x,y)\le l,\ i(x,y)\le m,\ j(x,y)=n$
All points, boundary piece 8 $\{g(x,y)\le k,\ h(x,y)=l,\ i(x,y)=m,\ j(x,y)\le n\}$		$g(x,y)\le k,\ h(x,y)=l,\ i(x,y)=m,\ j(x,y)\le n$
All points, boundary piece 9 $\{g(x,y)\le k,\ h(x,y)=l,\ i(x,y)\le m,\ j(x,y)=n\}$		$g(x,y)\le k,\ h(x,y)=l,\ i(x,y)\le m,\ j(x,y)=n$
All points, boundary piece 10 $\{g(x,y)\le k,\ h(x,y)\le l,\ i(x,y)=m,\ j(x,y)=n\}$		$g(x,y)\le k,\ h(x,y)\le l,\ i(x,y)=m,\ j(x,y)=n$

Sample setup ⑧ (3 variables, 1 inequality)

Global max/min of $f(x,y,z)$ on compact domain $\boxed{\{g(x,y,z)\le k\}}$



Sample domain

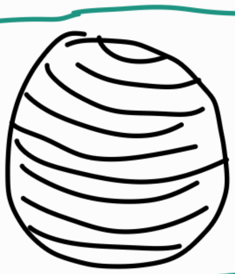
$\{x^2+y^2+z^2\le 1\}$

(Inside of a sphere)

Types		Equations to solve for (x,y,z)
Critical points, original domain		$\nabla f(x,y,z)=\langle 0,0,0\rangle$, $g(x,y,z)\le k$
Lagrange critical points Boundary piece ① $\{g(x,y,z)=k\}$	A	$\nabla g(x,y,z)=\langle 0,0,0\rangle$, $g(x,y,z)=k$
	B	$\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$ $g(x,y,z)=k$

Sample setup ⑨ (3 variables, 1 equality)

Global max/min of $f(x,y,z)$ on compact domain $\boxed{\{g(x,y,z)=k\}}$



Sample domain

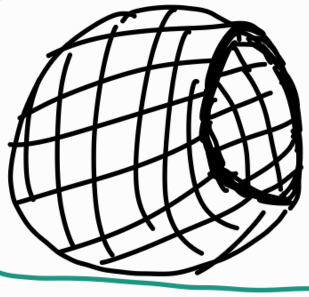
$\{x^2+y^2+z^2=1\}$

(Shell of a sphere)

Types		Equations to solve for (x,y,z)
Lagrange critical points Original domain	A	$\nabla g(x,y,z)=\langle 0,0,0\rangle$, $g(x,y,z)=k$
	B	$\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$ $g(x,y,z)=k$

Sample setup 10 (3 variables, 1 inequality, 1 equality)

Global max/min of $f(x,y,z)$ on compact domain $\boxed{\{g(x,y,z) \le k,\ h(x,y,z) = \ell\}}$



Sample domain

$\{x^2+y^2+z^2=1,\ x+y+z \le 1\}$

(Sphere shell, cut open)

Type		Equations to solve for (x,y,z)
Lagrange critical points original domain	A	$\nabla h(x,y,z) = \langle 0,0,0 \rangle$ $g(x,y,z) \le k,\quad h(x,y,z) = \ell$
	B	$\nabla f(x,y,z) = \lambda \nabla h(x,y,z)$ $g(x,y,z) \le k,\quad h(x,y,z) = \ell$
Lagrange critical points Boundary piece 1 $\{g(x,y,z) = k,\ h(x,y,z) = \ell\}$	A_1	$\nabla g(x,y,z) = \langle 0,0,0 \rangle$ $g(x,y,z) = k,\quad h(x,y,z) = \ell$
	A_2	$\nabla h(x,y,z) = \langle 0,0,0 \rangle$ $g(x,y,z) = k,\quad h(x,y,z) = \ell$
	B	$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$ $g(x,y,z) = k,\quad h(x,y,z) = \ell$

Sample setup (11) (3 variables, 2 equalities)

Global max/min of $f(x,y,z)$ on compact domain $\boxed{\{g(x,y,z)=k,\ h(x,y,z)=\ell\}}$

Sample domain

$\{x^2+y^2+z^2=1,\ x+y+z=1\}$

(Curve in 3D)

Types		Equations to solve for (x,y,z)
Lagrange critical points original domain	A$_1$	$\nabla g(x,y,z)=\langle 0,0,0\rangle$ $g(x,y,z)=k,\ h(x,y,z)=\ell$
	A$_2$	$\nabla h(x,y,z)=\langle 0,0,0\rangle$ $g(x,y,z)=k,\ h(x,y,z)=\ell$
	B	$\nabla f(x,y,z)=\lambda\nabla g(x,y,z)+\mu\nabla h(x,y,z$ $g(x,y,z)=k,\ h(x,y,z)=\ell$